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Jack Nachamkin

los alamos scientific laboratory
of the University of California
LOS ALAMOS, NEW MEXICO 87544

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THE INVARIANT DECOMPOSITION OF THE EQUATIONS GOVERNING HYDRODYNAMIC FLOW AND THERMAL CONDUCTIVITY

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ABSTRACT

The equations of hydrodynamics and thermal conductivity are spectrally decomposed using the properties of the (2L + 1) representations of the rotation group in three dimensions. The decomposition and subsequent set of nonlinear partial differential equations is derived by using the properties of the spherical harmonics, $Y_{\ell,m}$.

The familiar spherical harmonics, $Y_{\ell,m}(\Omega)$, encountered in mathematical physics have the properties

$$\int Y^*_{\ell,m}(\Omega) Y_{\ell',m'}(\Omega)\, d\Omega = \delta_{\ell\ell'}\delta_{mm'}, \qquad (1)$$

$$Y_{\ell m} Y_{\ell' m'} = \sum_L \left[\frac{(2\ell+1)(2\ell'+1)}{4\pi(2L+1)}\right]^{1/2} C^{\ell\ell' L}_{oo\,o} \times C^{\ell\ell' L}_{mm' M}\, Y_{L,M}\ \ell \qquad (2)$$

$$\nabla\left[\phi_{\ell,m}(r) Y_{\ell,m}\right] = \partial_\ell \phi_{\ell,m} \bar{T}_m^{(\ell+1,1)\ell} + \partial_{-\ell}\phi_{\ell,m} \times \bar{T}_m^{(\ell-1,1)} \qquad (3)$$

$$\nabla\cdot\left[\phi^\ell_{\ell',m'}(r)\bar{T}^{(\ell,1)\ell'}_{m'}\right] = -\left[\frac{(2\ell+1)}{(2\ell'+1)}\right]^{1/2} \times (\delta_{\ell',\ell+1}\partial_\ell\phi^\ell_{\ell'm'} + \delta_{\ell',\ell-1}\partial_{-\ell}\phi^\ell_{\ell',m'})\, Y_{\ell',m'} \qquad (4)$$

$$\nabla\times\left[\phi^\ell_{\ell',m'}(r)\bar{T}^{(\ell,1)\ell'}_{m'}\right] = (-1)^{\ell+\ell'}\left[3(2\ell+1)\right]^{1/2} \times \left[\partial_\ell\phi_{\ell m'}\begin{Bmatrix}1 & 1 & 1\\ \ell & \ell' & \ell+1\end{Bmatrix}\bar{T}^{(\ell+1,1)\ell'}_{m'} - \partial_{-\ell}\phi^\ell_{\ell',m'}\begin{Bmatrix}1 & 1 & 1\\ \ell & \ell' & \ell-1\end{Bmatrix}\times \bar{T}^{(\ell-1,1)\ell'}_{m'}\right]. \qquad (5)$$

In some of the equations written above the arguments Ω and $\underline{r}$ have been suppressed. It is implied that all the spherical harmonics in a given equation have the same solid-angle arguments. It will also be assumed that the functions $\phi^\ell_{\ell' m}$ and $\phi_{\ell m}$ are functions of the radial coordinate, r, only.

In equations 1-5: $\delta_{\ell,\ell'}$ is the familiar kronecker delta; ∇ is the usual gradient operator; $C^{\ell_1\ell_2\ell_3}_{m_1m_2m_3}$ is the Clebsch-Gordan coefficient encountered in coupling the quantum-mechanical angular momenta ℓ_1 and ℓ_2 to ℓ_3; $\begin{Bmatrix}j_1 & j_2 & j_5\\ j_4 & j_3 & j_6\end{Bmatrix}$ is the well-known 6-j coefficient; ∂_ℓ and $\partial_{-\ell}$ are the operators defined by

$$\partial_\ell \equiv -\left[\frac{\ell+1}{2\ell+1}\right]^{1/2} r^\ell \frac{\partial}{\partial r} r^{-\ell}, \qquad (6)$$

$$\partial_{-\ell} \equiv \left[\frac{\ell}{2\ell+1}\right]^{1/2} r^{-(\ell+1)}\frac{\partial}{\partial r} r^{\ell+1}; \qquad (7)$$

$\bar{T}^{(\ell_1,1)\ell_2}_{m_2}$ is the vector spherical harmonic encountered in the study of the electromagnetic field

$$\bar{T}^{(\ell_1,1)\ell_2}_{m_2} = \sum_{m_1 m} C^{\ell_1\ 1\ \ell_2}_{m_1\ m\ m_2} Y_{\ell_1,m_1}\xi_m, \qquad (8)$$

where ξ_m is the m-th component of the spherical unit vector.

We define now the hydrodynamic quantities:

$$\text{pressure} = p = \sum_{\ell,m} p_{\ell,m} Y_{\ell,m}, \qquad (9)$$

$$\text{specific volume} = V = \sum_{\ell,m} V_{\ell,m} Y_{\ell,m}, \qquad (10)$$

$$\text{density} = \rho = 1/V = \sum_{\ell,m} \rho_{\ell,m} Y_{\ell,m} \qquad (11)$$

$$\text{specific internal energy} = \mathcal{E} = \sum_{\ell,m} \mathcal{E}_{\ell,m} Y_{\ell,m}, \qquad (12)$$

$$\text{velocity} = \vec{U} = \Sigma\, U^{\ell}_{\ell',m'}\; \overline{T}^{(\ell,1)\ell'}_{m'} . \tag{13}$$

In Lagrangian coordinates, equations of hydrodynamics and thermal conductivity are

$$pV = \frac{2}{3}\mathcal{E} \text{ or } p = \frac{2}{3}\rho\mathcal{E}, \tag{14}$$

$$\frac{\partial\vec{u}}{\partial t} = -V\nabla p, \tag{15}$$

$$\frac{\partial\mathcal{E}}{\partial t} = -p\frac{\partial V}{\partial t} + V\nabla\cdot K\nabla T, \tag{16}$$

$$\frac{\partial\rho}{\partial t} = -\rho\nabla\cdot\vec{U} \text{ or } \frac{\partial V}{\partial t} = V\nabla\cdot\vec{U}. \tag{17}$$

Equation (14) is the equation of state of an ideal gas, whether or not it is degenerate. The quantity T is the temperature of the gas, $T = \Sigma T_{\ell m} Y_{\ell m}$, and K is the thermal conductivity. By patient manipulation we arrive at the following equations for the quantities in Eqs. (9-13). (Dots denote time differentiation.)

$$\dot{U}^{\ell}_{\ell_1,m_1} = (-1)^{\ell_1-\ell} \sum_{\substack{\ell_2\ell_3\ell_4\\ m_2 m_3}} V_{\ell_2,m_2}\left[\frac{(2\ell_2+1)(2\ell_4+1)}{4\pi}\right]^{1/2}$$

$$C^{\ell_2\ell_4\ell}_{m_2m_4m_1} \times \left[C^{\ell_2\,\ell_3+1\,\ell}_{o\;\;o\;\;o}\begin{Bmatrix}\ell_1 & 1 & \ell\\ \ell_3+1 & \ell_2 & \ell_4\end{Bmatrix}\partial_{\ell_3}\rho_{\ell_3,m_3} - \right.$$

$$\left. C^{\ell_2\,\ell_3-1\,\ell}_{o\;\;o\;\;o}\begin{Bmatrix}\ell_1 & 1 & \ell\\ \ell_3-1 & \ell_2 & \ell_4\end{Bmatrix}\partial_{-\ell_3}\rho_{\ell_3,m_3}\right]. \tag{18}$$

$$\dot{\mathcal{E}}_{\ell,m}\Big|_{Hydro} = \Sigma\rho_{\ell_1,m_1}\dot{V}_{\ell_2,m_2}\left[\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell+1)}\right]^{1/2}$$

$$C^{\ell_1\ell_2\ell}_{m_1m_2m}\, C^{\ell_1\ell_2\ell}_{o\;o\;o} \tag{19a}$$

$$\dot{\mathcal{E}}_{\ell,m}\Big|_{Thermal} = \sum_{\substack{\ell''m''\ell'm'\\ \ell\, m\, L\, L''\\ \alpha\,\beta}} \left[\frac{(2L+1)(2\ell''+1)}{4\pi(2L''+1)}\right]^{1/2} V_{\ell''m''} \times$$

$$A^{\ell,m,\ell',m',L}_{\alpha,\beta}\, C^{\ell''\,L+\beta\,L''}_{o\;\;o\;\;o} \times C^{\ell''\,L+\beta\,L''}_{m''\,M'\,M''}\,\partial_{\alpha\ell}K_{\ell,m}\,\partial_{\beta\ell'}$$

$$\times\, T_{\ell',m'}\, Y_{L'',M''}. \tag{19b}$$

$$\dot{\mathcal{E}}_{\ell,m} = \dot{\mathcal{E}}_{\ell,m}\Big|_{Hydro} + \dot{\mathcal{E}}_{\ell,m}\Big|_{Thermal} \tag{19c}$$

where

$$A^{\ell,m,\ell',m',L}_{\alpha,\beta} = (-1)^{\ell}\left[\frac{(2\ell+1)(2\ell'+1)(2L+1)}{4\pi}\right]^{1/2}$$

$$C^{\ell L\,\ell'+\alpha}_{o\;o\;\;o}\, C^{\ell'\ell\,L+\beta}_{m'm\;\;M}\begin{Bmatrix}L & \ell'+1 & \ell\\ \ell' & L+1 & 1\end{Bmatrix}. \tag{20}$$

Quantities α and β in equations (19) and (20) can both assume the values of +1 and -1 independently. The equations of continuity are

$$\dot{\rho}_{\ell,m} = -\sum_{\ell_1,m_1,\ell_2,m_2,\ell_3,\alpha}\left[\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell+1)}\right]^{1/2}$$

$$C^{\ell_1\ell_2\ell}_{m_1m_2m}\, C^{\ell_1\ell_2\ell}_{o\;o\;o}\, V_{\ell_1,m_1}\,\partial_{\alpha\ell_3}U^{\ell_3}_{\ell,+\alpha,m_2} \tag{21a}$$

or

$$\dot{V} = \sum_{\ell_1,m_1,\ell_2,m_2,\ell_3,\alpha}\left[\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell+1)}\right]^{1/2}$$

$$C^{\ell_1\ell_2\ell}_{m_1m_2m}\, C^{\ell_1\ell_2\ell}_{o\;o\;o}\, V_{\ell_1,m_1}\,\partial_{\alpha\ell_3}\, U^{\ell_3}_{\ell_3+\alpha,m_2} . \tag{21b}$$

From hindsight obtained gained from observing the above equations we see that angular momentum, linear momentum, and parity are conserved by our fluid. One notes that the ∇ operator has a pole at r= 0 so that perturbation expansions near r=0 are generally invalid. The striking fact, however, derives from the conservation of angular momentum. First note that any $\overline{T}^{(\ell_1,1)\ell_2}_{m_2}$, $\ell_1>0$ represents a spin [see equation (5)]. Equations (18), (19), and (20) tell us that any $Y_{\ell,m}$'s in the velocity field will under spherical compression

- Be damped to $Y_{z,m}$'s such that:
- All the vorticial and total angular momentum will be in special $\overline{T}^{(\ell,1)\ell}$'s.
- The entropy of these $\overline{T}^{(\ell,1)\ell}$'s will increase very slowly compared to compression of a $Y_{o,o}$. This is because the $\overline{T}^{(\ell,1)\ell}$'s represent vortices. The vortices, to conserve vorticial spin shift spherically applied compressional energy mostly into rotational motion, not entropy, no matter how suddenly the compression is applied. This fact may lead to "freezing in" a low temperature profile while producing a fluid-flow velocity greatly in excess of the corresponding mean-square velocity corresponding to the temperature profile. These effects may be a possible explanation of some of the laser-pellet experiments performed by Gene McCall[1] and his coworkers in L division. Note also that compressing the outside of a vortex does not necessarily increase the density at the center. Thus, after a certain point of vorticial compression is achieved, the fluid velocities may be enormous but the central compression will decrease with increasing applied force.)

Perturbation expansion of equations (9-13) are being investigated as to what possible symmetries might arise in a compression of a fluid.

We note in passing that a new philosophy may have to be applied to code production, especially where three-dimensional processes are involved. High, yet subtle, symmetries may generally be expected which can absorb a great deal of energy, yet can neither substantially raise their entropy nor couple this energy to a different mode. Vortices belong to this class. A code which has the smallest amount of coupling between a curl-free and vorticial state will invariably transfer the enrgy to the streamline-flow state. Hydro codes which do not obey the law of exact angular momentum conservation are subject to sterner criticism in some respects and in certain circumstances than codes with no exact conservation of energy built in.

REFERENCES

1. McCall, G. H., Bull. Am. Phys. Soc., Series II, Vol. 18, No. 10, p. 1255 (1973).